Chapter 5
Transient Analysis

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Complete response = Transient response + Steady-state response
Time Constant
First order and Second order Differential Equation

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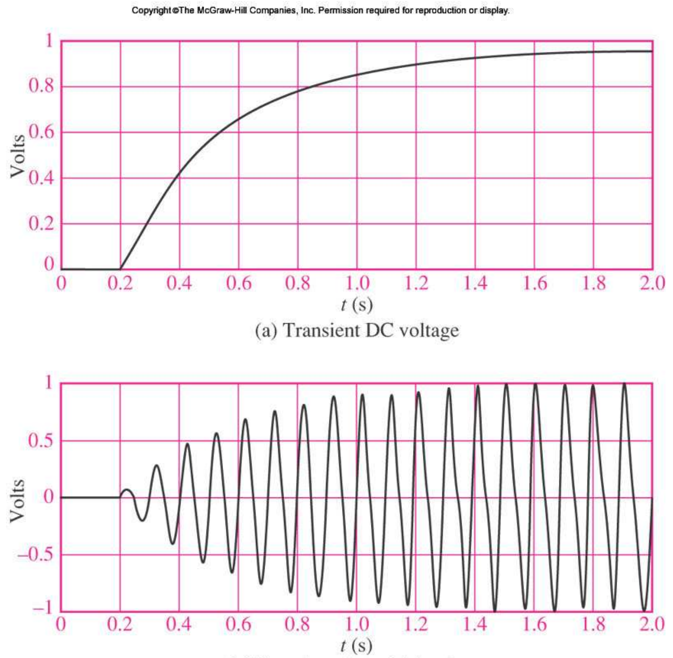
Transient Analysis

- The difference of analysis of circuits with energy storage elements (inductors or capacitors) & time-varying signals with resistive circuits is that the equations resulting from KVL and KCL are now <u>differential equations</u> rather than <u>algebraic linear equations</u> resulting from the resistive circuits.

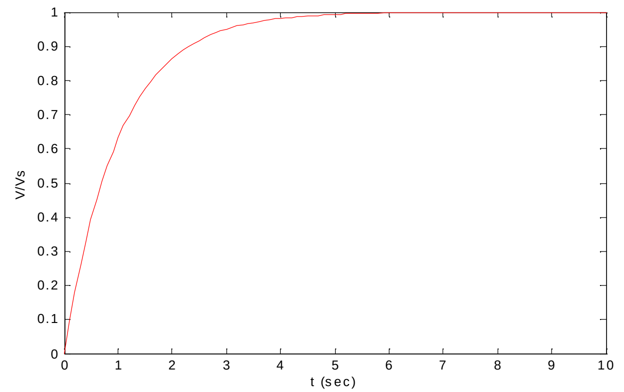


(a) Transient DC voltage

(b) Transient sinusoidal voltage

- Transient region: the region where the signals are highly dependent on time. (temporary)
  - No voltage or current sources
  - Transient Analysis
- Steady-state region: the region where the signals are not time dependent (time rate of change of signals is equal to zero) or periodic.
  - Constant signals
  - Sinusoidal signals

$$\frac{d(\ )}{dt} = 0$$



Solution of Ordinary Differential Equation

- Transient solution (x_N) is a solution of the homogeneous equation: transient (natural) response. -> <u>temporary behavior without the source.</u>
- Steady-state (particular) solution (x_F) is a solution <u>due to the source</u>: steady-state (forced) response.

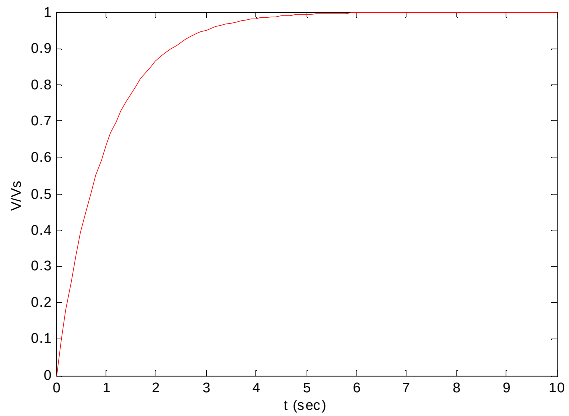
$$\frac{dx}{dt} + x = V_s$$

$$\frac{dx_N}{dt} + x_N = 0$$

- Complete response = transient (natural) response + steady-state (forced) response -> $x = x_N + x_F$

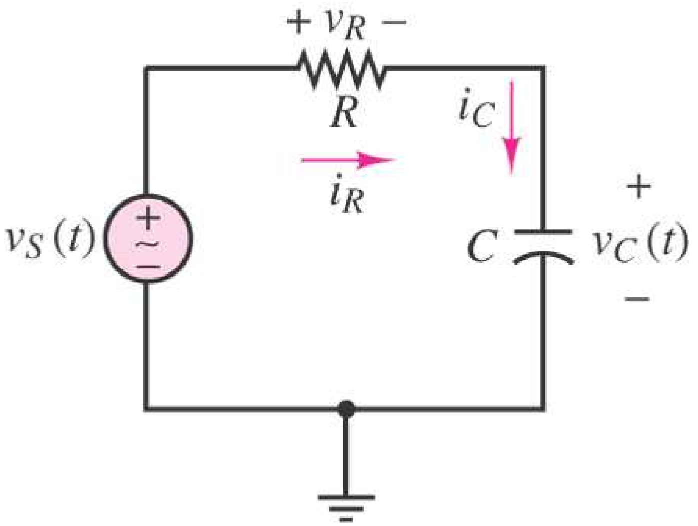
$$\frac{dx_F}{dt} + x_F = V_s$$

- First order: The largest order of the differential equation is the first order.
  - *RL* or *RC* circuit.
- Second order: The largest order of the differential equation is the second order.
  - *RLC* or *LC* circuit.



Writing Differential Equations

- Key laws: KVL & KCL for capacitor voltages or inductor currents



$$\text{KCL}: i_R = i_C \rightarrow \frac{v_R}{R} = i_C$$

$$\text{KVL}: -v_S + v_R + v_C = 0 \rightarrow -v_S + i_R R + v_C = 0$$

$$i_C R + v_C(t=0) + \int_0^t \frac{i_C(t')}{C} dt' = v_S$$

$$\frac{di_C}{dt} R + \frac{i_C}{C} = \frac{dv_S}{dt} \rightarrow \frac{di_C}{dt} + \frac{i_C}{RC} = \frac{dv_S}{Rdt} : \text{Differential equation for } i_C$$

$$\frac{v_R}{R} = i_C = C\frac{dv_C}{dt} = \frac{v_S - v_C}{R} \rightarrow \frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{v_S}{RC} : \text{Differential equation for } v_C$$

$$a_1 \frac{dx(t)}{dt} + a_0 x(t) = b_0 f(t)$$

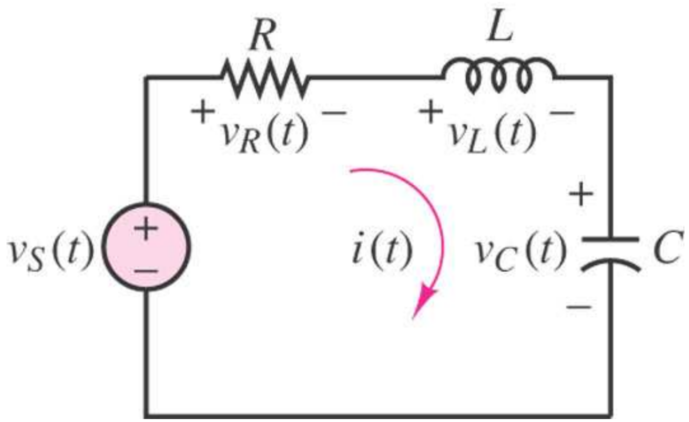
where $x(t)$ represents the capacitor voltage or the inductor current and

the constants a_1, a_0, and b_0 represents combinations of circuit element parameters.

$\rightarrow$ First - order linear ordinary differential equation

Writing Differential Equations (cont.)

- Key laws: KVL & KCL



$$\text{KCL}: i_R = i_C = i_L = i$$

$$\text{KVL}: -v_S + v_R + v_C + v_L = 0 \rightarrow -v_S + i_R R + v_C + v_L = 0$$

$$iR + v_C(t=0) + \int_0^t \frac{i(t')}{C}dt' + L\frac{di}{dt} = v_S$$

$$\frac{di}{dt}R + \frac{i}{C} + L\frac{d^2 i}{dt^2} = \frac{dv_S}{dt} \rightarrow \frac{d^2 i}{dt^2} + \frac{Rdi}{Ldt} + \frac{i}{LC} = \frac{dv_S}{Ldt} : \text{Differential equation for } i$$

$$\frac{v_R}{R} = i_C = C\frac{dv_C}{dt} = \frac{v_S - v_C - v_L}{R} \rightarrow C\frac{dv_C}{dt} = \frac{v_S}{R} - \frac{v_C}{R} - \frac{1}{R}\left(L\frac{d}{dt}C\frac{dv_C}{dt}\right)$$

$$RC\frac{dv_C}{dt} = v_S - v_C - LC\left(\frac{d^2 v_C}{dt^2}\right)$$

$$LC\left(\frac{d^2 v_C}{dt^2}\right) + RC\frac{dv_C}{dt} + v_C = v_S : \text{Differential equation for } v_C$$

$$a_2\frac{d^2 x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_0 x(t) = b_0 f(t) \rightarrow \text{Second-order linear ordinary differential equation}$$

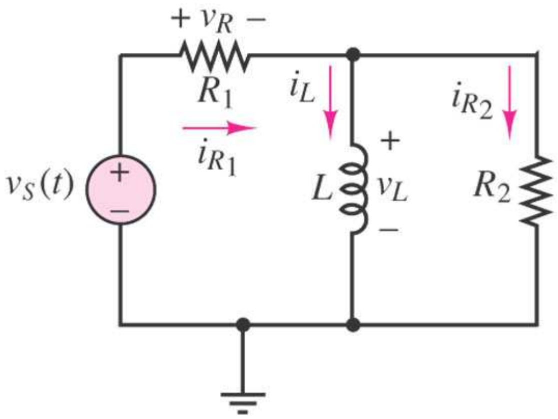
where $x(t)$ represents the capacitor voltage or the current and

the constants a_2, a_1, a_0, and b_0 represents combinations of circuit element parameters.

$$\frac{a_2}{a_0}\frac{d^2 x(t)}{dt^2} + \frac{a_1}{a_0}\frac{dx(t)}{dt} + x(t) = \frac{b_0}{a_0} f(t) \rightarrow \frac{1}{\omega_n^2}\frac{d^2 x(t)}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx(t)}{dt} + x(t) = K_S f(t)$$

where the constants $\omega_n = \sqrt{a_0/a_2}$, $\zeta = (a_1/2)\sqrt{1/a_0 a_2}$ and $K_S = b_0/a_0$ termed the natural frequency, the damping ratio, and the DC gain, respectively.

Examples of Writing Differential Equations



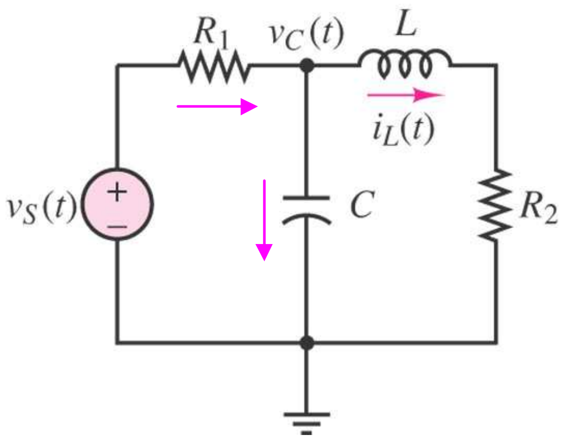
$$\text{KCL}: i_{R_1} = i_L + i_{R_2} \rightarrow \frac{v_R}{R} = i_L + i_{R_2}$$

$$\text{KVL}: -v_S + v_R + v_L = 0 \rightarrow v_R = v_S - v_L$$

$$\frac{v_R}{R} = i_L + i_{R_2} \rightarrow \frac{v_S - v_L}{R} = i_L(t=0) + \int_0^t \frac{v_L(t')}{L} dt' + \frac{v_L}{R}$$

$$v_S - v_L = Ri_L(t=0) + \int_0^t \frac{Rv_L(t')}{L} dt' + v_L \rightarrow v_S = Ri_L(t=0) + \int_0^t \frac{Rv_L(t')}{L} dt' + 2v_L$$

$$\frac{dv_S}{dt} = \frac{R}{L} v_L + \frac{2dv_L}{dt} \rightarrow 2\frac{dv_L}{dt} + \frac{R}{L} v_L = \frac{dv_S}{dt} : \text{Differential equation for } v_L$$



$$\text{KCL}: i_{R_1} = i_C + i_L$$

$$\text{KVL}: -v_S + v_{R_1} + v_C = 0 \rightarrow v_S = v_{R_1} + v_C$$

$$-v_C + v_{R_2} + v_L = 0 \rightarrow v_C = v_{R_2} + v_L = L\frac{di_L}{dt} + i_L R_2$$

$$v_{R_1} = i_{R_1} R_1 = (i_C + i_L) R_1 = \left(C\frac{dv_C}{dt} + i_L \right) R_1 = \left(C\frac{d}{dt}\left(L\frac{di_L}{dt} + i_L R_2 \right) + i_L \right) R_1 = \left(LC\frac{d^2 i_L}{dt^2} + R_2 C\frac{di_L}{dt} + i_L \right) R_1$$

$$v_S = v_{R_1} + v_C = \left(LC\frac{d^2 i_L}{dt^2} + R_2 C\frac{di_L}{dt} + i_L \right) R_1 + L\frac{di_L}{dt} + i_L R_2 \rightarrow$$

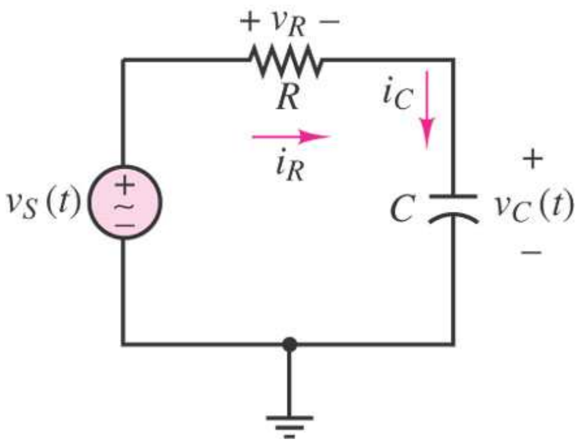
$$v_S = R_1 LC\frac{d^2 i_L}{dt^2} + R_1 R_2 C\frac{di_L}{dt} + R_1 i_L + L\frac{di_L}{dt} + i_L R_2 \rightarrow$$

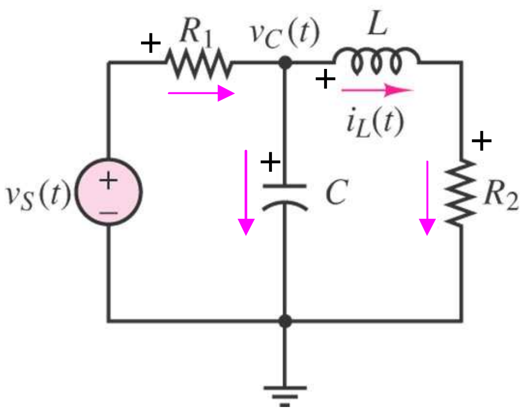
$$R_1 LC\frac{d^2 i_L}{dt^2} + (R_1 R_2 C + L)\frac{di_L}{dt} + (R_1 + R_2) i_L = v_S : \text{Differential equation for } i_L$$

DC steady state solution: Final Condition

- Steady state solution due to AC (sinusoidal waveforms) is in Chap. 6 (frequency response).
- DC steady state solution: response of a circuit that have been connected to a DC source for a long time or response of a circuit long after a switch has been activated.
  - All the time derivatives are equal to zero at the steady state.

- Capacitors: insulators (very large resistances) are inside the capacitors.
- Inductors: Induction works only when the change in electric fields happens.





$$i_C(t) = C\frac{dv_C(t)}{dt} \to 0 \text{ as } t \to \infty$$

$$v_L(t) = L\frac{di_L(t)}{dt} \to 0 \text{ as } t \to \infty$$

At DC steady state,

all capacitors behave as open circuits

and all inductors hehave as short circuits.

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{v_S}{RC}$$

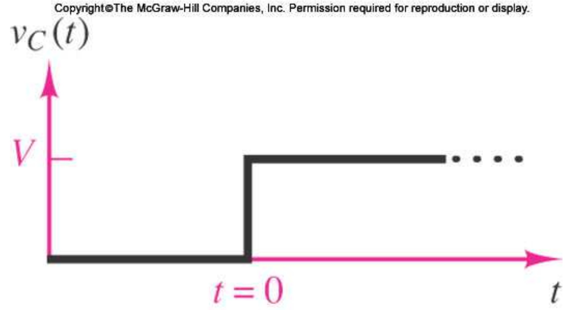
$v_C = v_S$ at the steady state

$$R_1LC\frac{d^2 i_L}{dt^2} + (R_1R_2C + L)\frac{di_L}{dt} + (R_1 + R_2)i_L = v_S$$

$i_L = \dfrac{v_S}{(R_1 + R_2)}$ at the steady state

DC steady state solution: Initial Condition

- Initial condition: response of a circuit before a switch is first activated.
  - Since power equals energy per unit time, finite power requires continuous change in energy.
- Primary variables: capacitor voltages and inductor currents-> energy storage elements $W_L(t) = \frac{1}{2} L i_L^2(t) \quad W_C(t) = \frac{1}{2} C v_C^2(t)$
  - Capacitor voltages and inductor currents cannot change instantaneously but should be continuous. -> continuity of capacitor voltages and inductor currents
  - The value of an inductor current or a capacitor voltage just prior to the closing (or opening) of a switch is equal to the value just after the switch has been closed (or opened).



$$v_C(t = 0^-) = v_C(t = 0^+)$$
$$i_L(t = 0^-) = i_L(t = 0^+)$$

where the notation 0^- signifies "just before $t = 0$" and

0^+ signifies "just after $t = 0$"

Discontinuous of capacitor voltage
-> infinite power at t=0.

First Order Response

- First-order circuit: one energy storage element + one energy loss element (e.g. *RC* circuit, *RL* circuit)

- Procedures
  - Write the differential equation of the circuit for $t=0^+$, that is, immediately after the switch has changed. The variable x(*t*) in the differential equation will be either a capacitor voltage or an inductor current. You can reduce the circuit to Thevenin or Norton equivalent form.
  - Identify the initial conditions $x(t=0^+)$ [= $x(t=0^-)$] and final conditions $x(t=\infty)$.
  - Solve the differential equation.
  - Write the complete solution for the circuit in the form.

$$x(t) = x(t=\infty) + \left[x(t=0) - x(t=\infty)\right]\exp(-t/\tau)$$

- The time constant (τ) is a measure of <u>how fast</u> capacitor voltages or inductor currents <u>react</u> to the input (voltage or current source). It is a period of time during which capacitor voltages or inductor currents change by 63.2% to get to the steady state.

$$\frac{\left[x(t=\tau) - x(t=0)\right]}{\left[x(t=\infty) - x(t=0)\right]} = 1 - e^{-1} = 0.632$$

First Order Response (cont.)

- First-order circuit: one energy storage element + one energy loss element (e.g. *RC* circuit, *RL* circuit)

$$a_1 \frac{dx(t)}{dt} + a_0 x(t) = b_0 f(t) \rightarrow \frac{a_1}{a_0} \frac{dx(t)}{dt} + x(t) = \frac{b_0}{a_0} f(t) \rightarrow \tau \frac{dx(t)}{dt} + x(t) = K_S f(t)$$

where $\tau = a_1 / a_0$ and $K_S = b_0 / a_0$ termed the time constant and DC gain, respectively.

Natural Response

$$\tau \frac{dx_N(t)}{dt} + x_N(t) = 0 \rightarrow \frac{dx_N(t)}{dt} = \frac{-x_N(t)}{\tau} \rightarrow x_N(t) = x_0 e^{-t/\tau} \text{ where } x_0 \text{ is a constant.}$$

Forced Response due to DC (where $f(t) = F$): $\frac{dx_F(t)}{dt} \rightarrow 0$

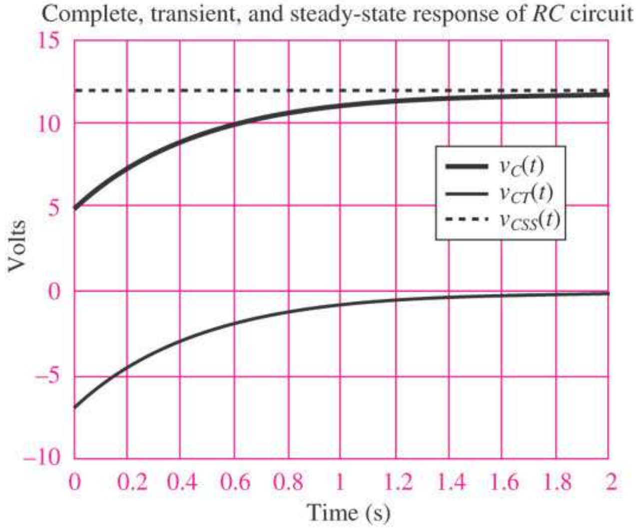
$$\tau \frac{dx_F(t)}{dt} + x_F(t) = K_S F \quad t \geq 0 \rightarrow x_F(t) = K_S F \quad t \geq 0$$

Complete Response

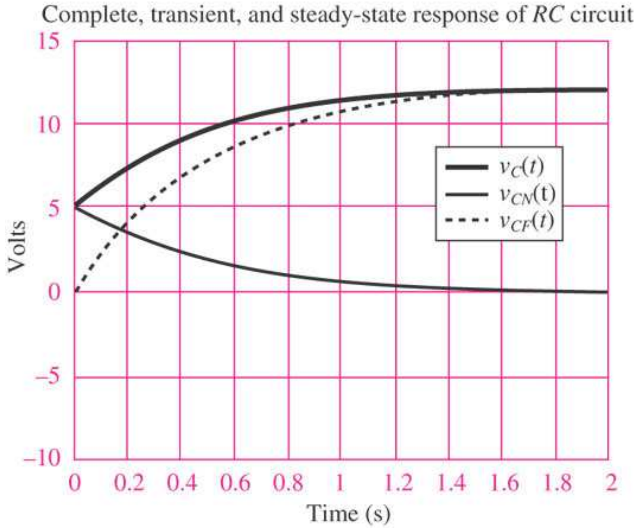
$$x(t) = x_N(t) + x_F(t) = x_0 e^{-t/\tau} + x(t = \infty) = x_0 e^{-t/\tau} + K_S F \text{(for DC)}$$

$$x(t = 0) = x_0 + x(t = \infty) \rightarrow x_0 = x(t = 0) - x(t = \infty) \text{ for } t \geq 0$$

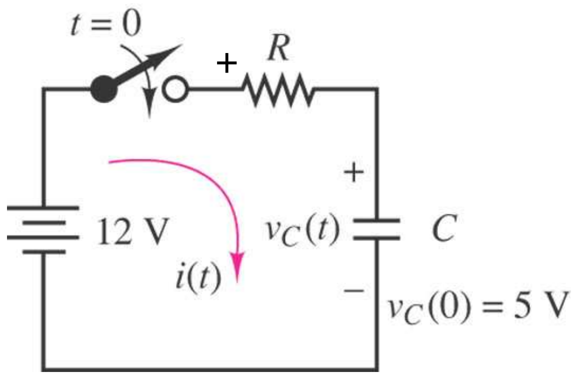
Example: First Order Response 1



(a)



(b)



Step1: KCL: $i_R = i_C \rightarrow \frac{v_R}{R} = i_C$

KVL: $-v_S + v_R + v_C = 0 \rightarrow -v_S + i_R R + v_C = 0$

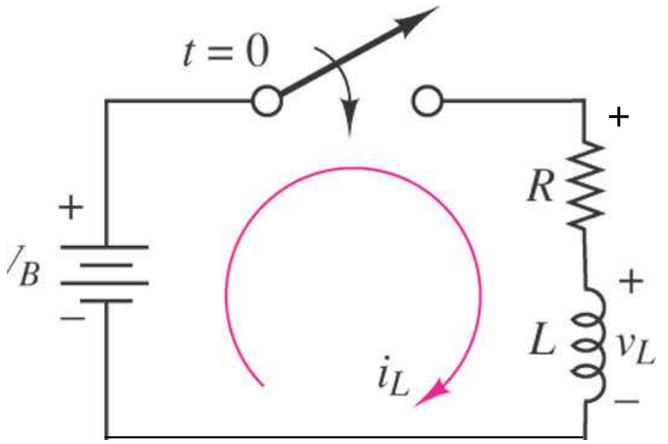
$$\frac{v_R}{R} = i_C = C\frac{dv_C}{dt} = \frac{v_S - v_C}{R} \rightarrow RC\frac{dv_C}{dt} + v_C = v_S \quad t > 0 \rightarrow \tau\frac{dx(t)}{dt} + x(t) = K_S F$$

Step2: $v_C(t = 0^-) = 5\text{ V} = v_C(t = 0^+), v_C(t = \infty) = 12\text{V}(= v_S)$

Step3: $x = v_C, \tau = RC = 1\text{k}\Omega \times 470\mu\text{F} = 0.47, K_S = 1, F = v_S$

Step4: $v_C(t) = (v_C(t = 0) - v_C(t = \infty))e^{-t/\tau} + v_C(t = \infty) = 12 + (-7)e^{-t/0.47}$

Example: First Order Response 2



Step1: KCL : $i_R = i_L$

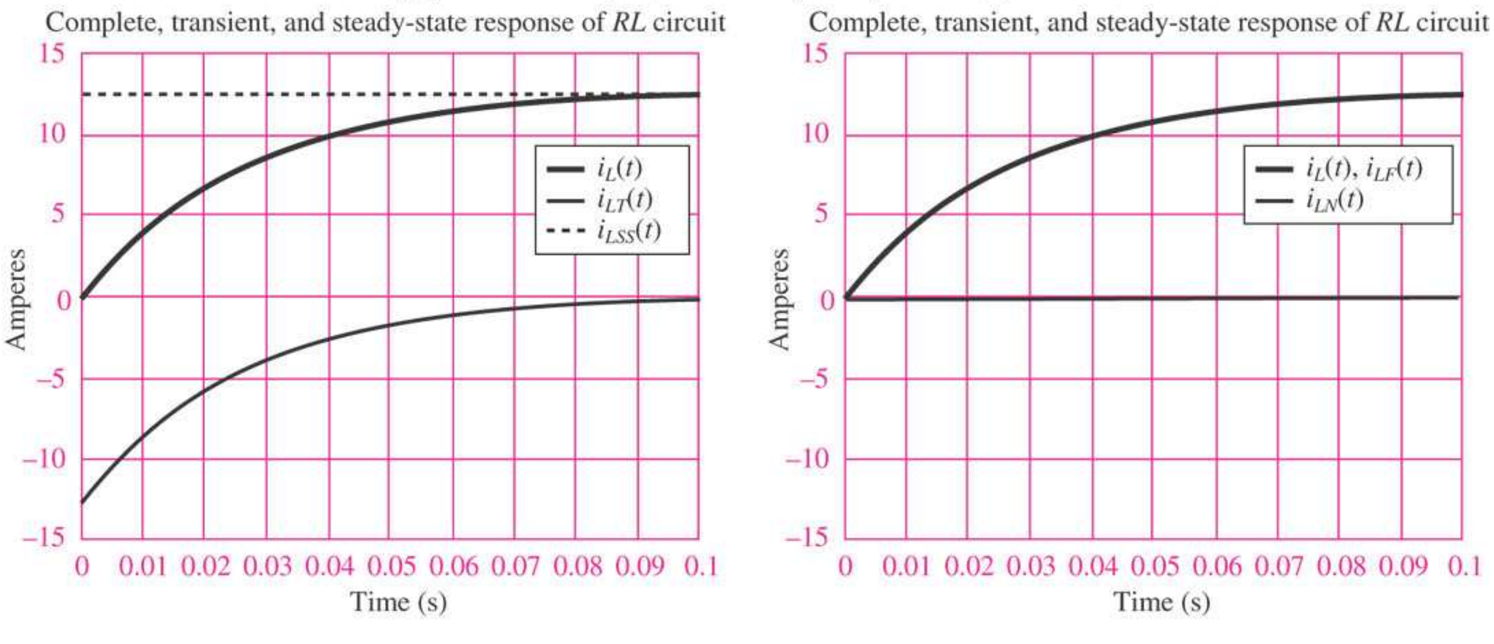
$$\text{KVL} : -v_B + v_R + v_L = 0 \rightarrow -v_B + i_L R + L\frac{di_L}{dt} = 0$$

$$\rightarrow \frac{L}{R}\frac{di_L}{dt} + i_L = \frac{v_B}{R} \quad t > 0 \rightarrow \tau\frac{dx(t)}{dt} + x(t) = K_S F$$

Step2: $i_L(t = 0^-) = 0\text{ A} = i_L(t = 0^+), i_L(t = \infty) = v_B/R = 12.5\text{A}$

Step3: $x = i_L, \tau = L/R = 0.1\text{H}/4\Omega = 0.025, K_S = 1/R, F = v_B$

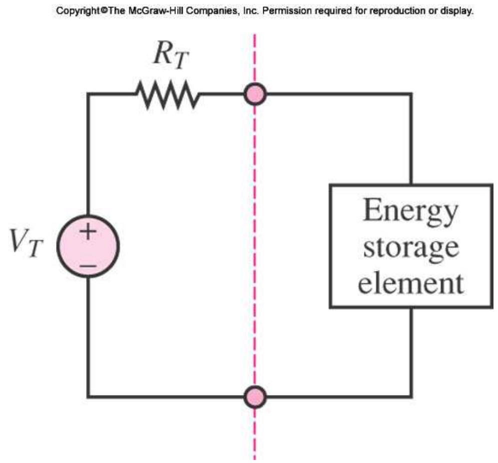
Step4: $i_L(t) = (i_L(t = 0) - i_L(t = \infty))e^{-t/\tau} + i_L(t = \infty) = 12.5 + (-12.5)e^{-t/0.025}$

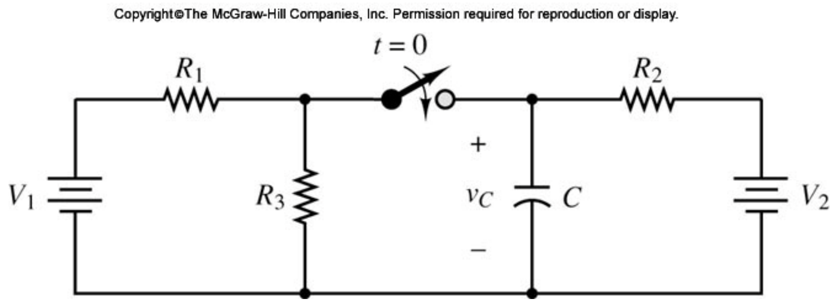


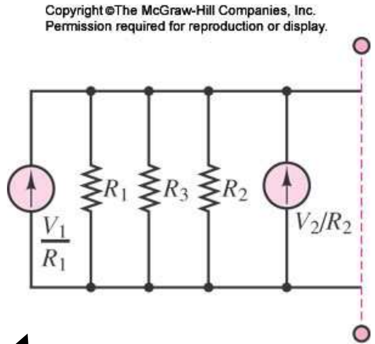
(a) (b)

First Order Transient Response Using Thevenin/Norton Theorem

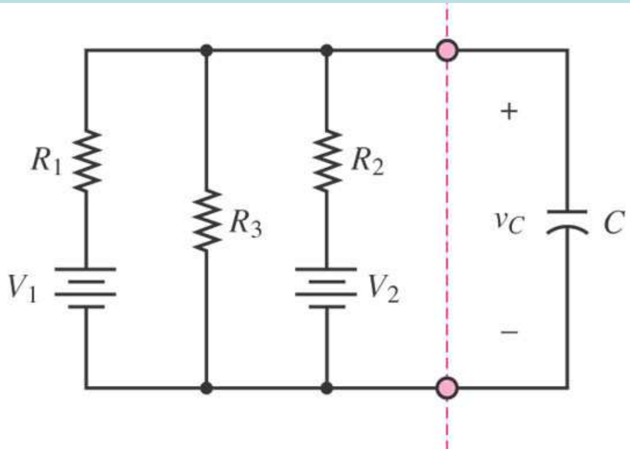
- One must be careful to determine the equivalent circuits before and after the switch changes position.
  - it is possible that equivalent circuit seen by the load before activating the switch is different from the circuit seen after closing the switch.

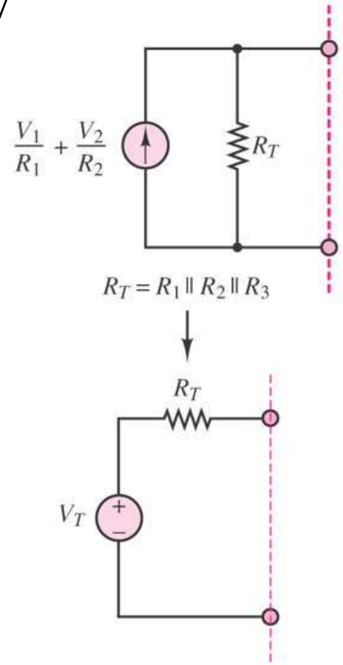


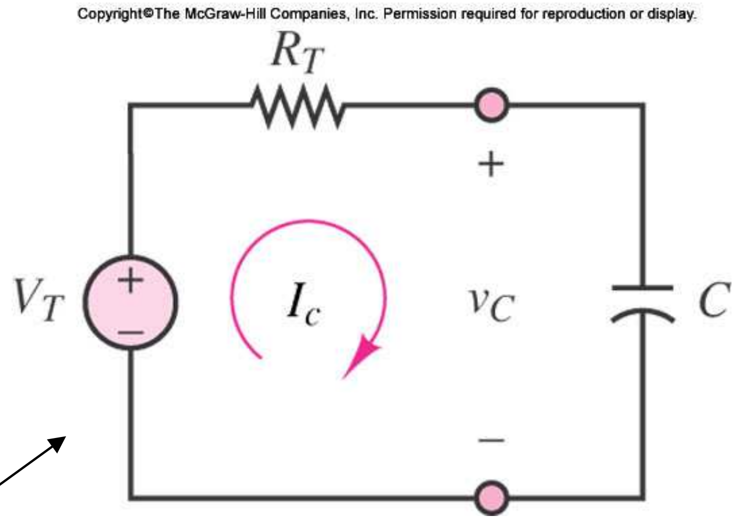




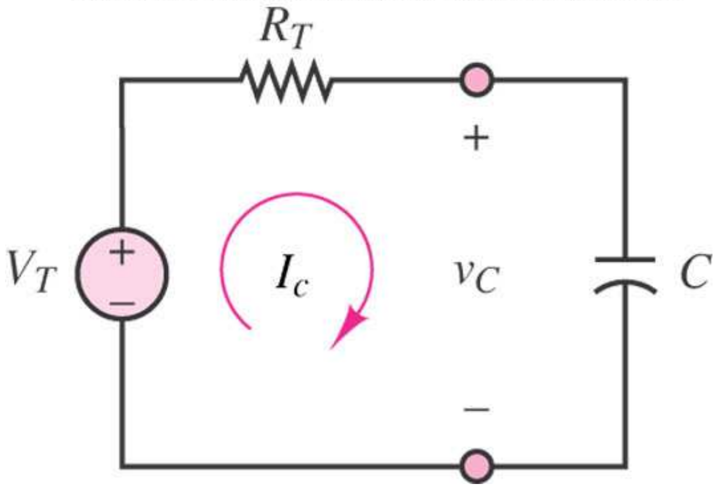
$$v_C(t)=V_2 \quad t\le 0 \qquad v_C\left(t=0^-\right)=V_2=v_C\left(t=0^+\right)$$







First Order Transient Response Using Thevenin/Norton Theorem (cont.)



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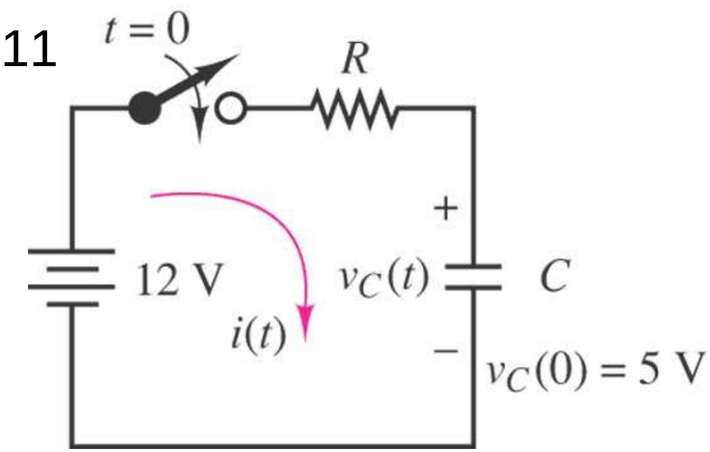


$$\text{Step1:}\quad R_T C \frac{dv_C}{dt} + v_C = V_T \quad t > 0 \rightarrow \tau \frac{dx(t)}{dt} + x(t) = K_S F$$

$$\text{Step2:}\quad v_C\left(t = 0^-\right) = V_2 = v_C\left(t = 0^+\right), v_C(t = \infty) = V_T$$

$$\text{Step3:}\quad x = v_C, \tau = R_T C, K_S = 1, F = V_T$$

$$\text{Step4:}\quad v_C(t) = \left(v_C(t = 0) - v_C(t = \infty)\right)e^{-t/\tau} + v_C(t = \infty) = \left(V_2 - V_T\right)e^{-t/\tau} + V_T$$

$$R_T = R_1 \parallel R_2 \parallel R_3 \qquad V_T = R_T\left(\frac{V_1}{R_1} + \frac{V_2}{R_2}\right)$$

$$\text{Step1:}\quad RC \frac{dv_C}{dt} + v_C = v_S \quad t > 0 \rightarrow \tau \frac{dx(t)}{dt} + x(t) = K_S F$$

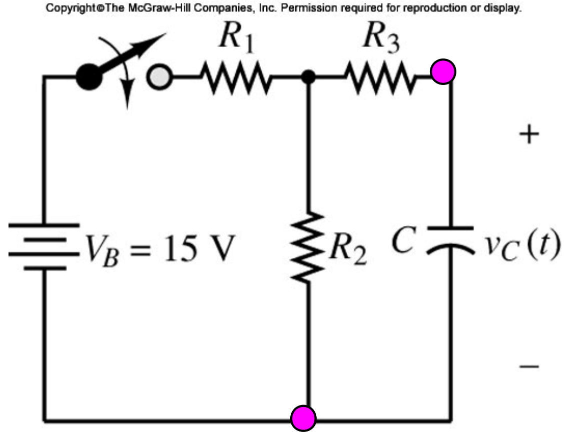
$$\text{Step2:}\quad v_C\left(t = 0^-\right) = v_C\left(t = 0^+\right), v_C(t = \infty) = v_S$$

$$\text{Step3:}\quad x = v_C, \tau = RC, K_S = 1, F = v_S$$

$$\text{Step4:}\quad v_C(t) = \left(v_C(t = 0) - v_C(t = \infty)\right)e^{-t/\tau} + v_C(t = \infty)$$

First Order Transient Response Using Thevenin/Norton Theorem (cont.)

Example 5.10



0 (closing) < t < 50 ms

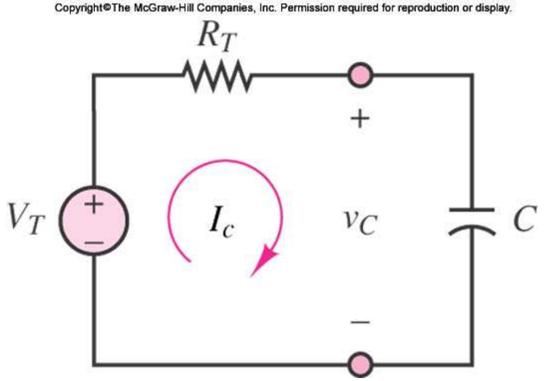
$$\text{Step1}: \; R_T C \frac{dv_C}{dt} + v_C = V_T \quad t > 0 \rightarrow \tau \frac{dx(t)}{dt} + x(t) = K_S F$$

$$\text{Step2}: \; v_C\left(t = 0^-\right) = 0 = v_C\left(t = 0^+\right), v_C(t = \infty) = V_T$$

$$\text{Step3}: \; x = v_C, \tau = R_T C, K_S = 1, F = V_T$$

$$\text{Step4}: \; v_C(t) = \left(v_C(t=0) - v_C(t=\infty)\right)e^{-t/\tau} + v_C(t=\infty) = \left(-V_T\right)e^{-t/\tau} + V_T$$

$$\boxed{R_T = \left(R_1 \parallel R_2\right) + R_3 \qquad V_T = \frac{R_2}{R_1 + R_2} V_B : \text{voltage divider}}$$



50 ms (open the switch again) < t

$$\text{Step1}: \; R_T C \frac{dv_C}{dt} + v_C = 0 \quad t > 0 \rightarrow \tau \frac{dx(t)}{dt} + x(t) = K_S F$$

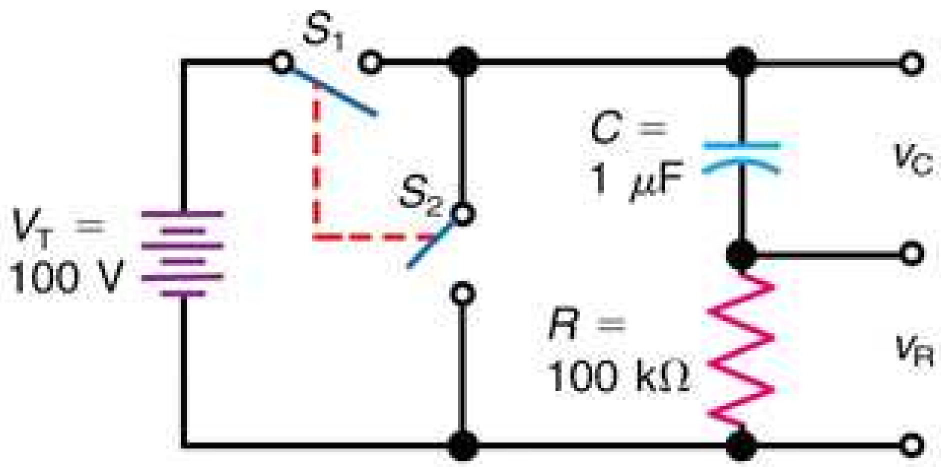
$$\text{Step2}: \; v_C\left(t = 0^-\right) = v_C(t = 50ms)(\text{from the solution above}) = v_C^* = v_C\left(t = 0^+\right), v_C(t = \infty) = 0$$

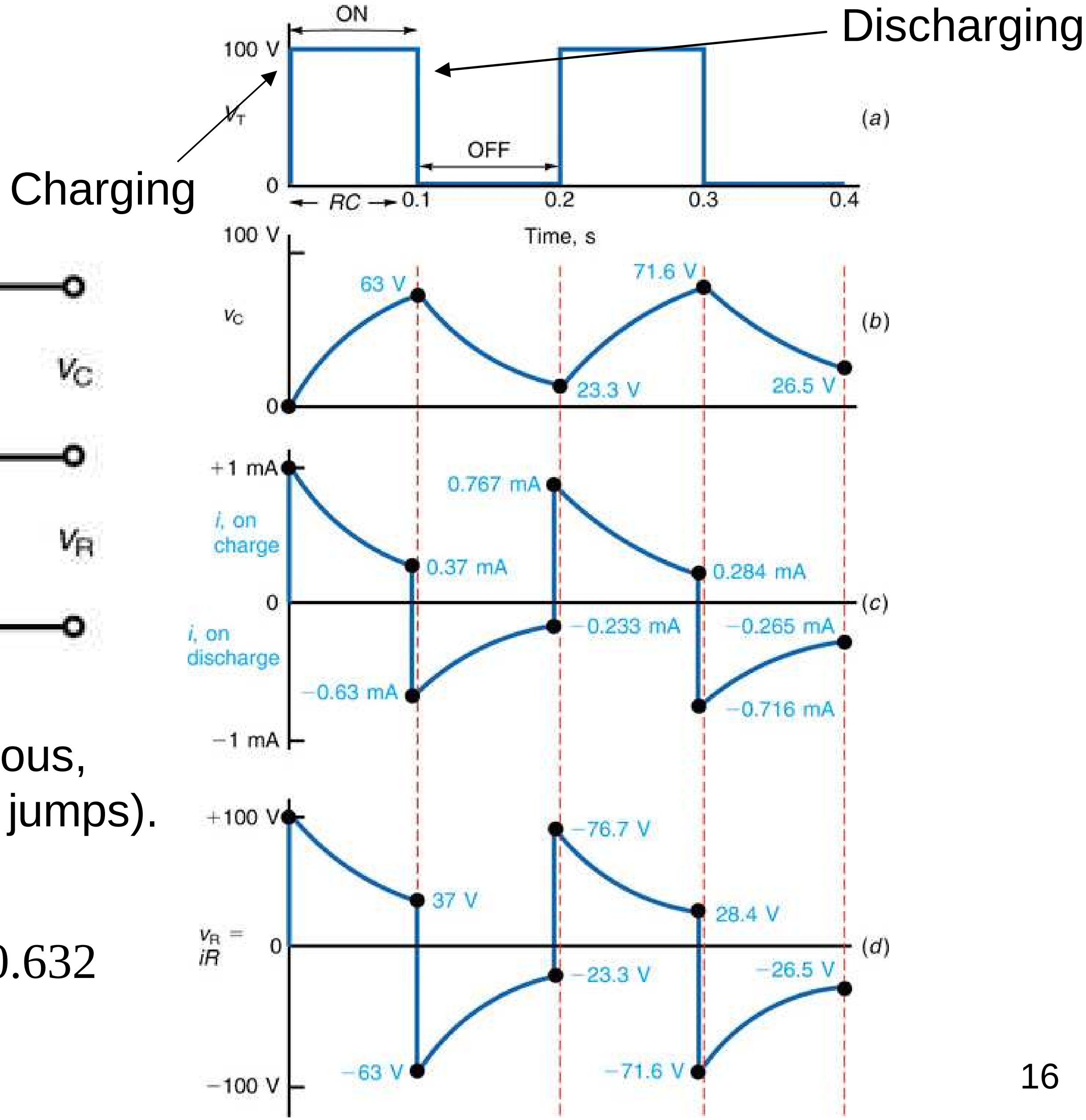
$$\text{Step3}: \; x = v_C, \tau = R_T C, K_S = 1, F = 0 \text{ where } \boxed{R_T = R_2 + R_3}$$

$$\text{Step4}: \; v_C(t) = \left(v_C(t=0) - v_C(t=\infty)\right)e^{-t/\tau} + v_C(t=\infty) = \left(v_C^*\right)e^{-t/\tau} \rightarrow v_C(t) = \left(v_C^*\right)e^{-(t-0.05)/\tau}$$

RC Charging & Discharging

Charging: S_1 closed & S_2 opened
Discharging: S_2 closed & S_1 opened
Time constant (τ = RC)=0.1 sec



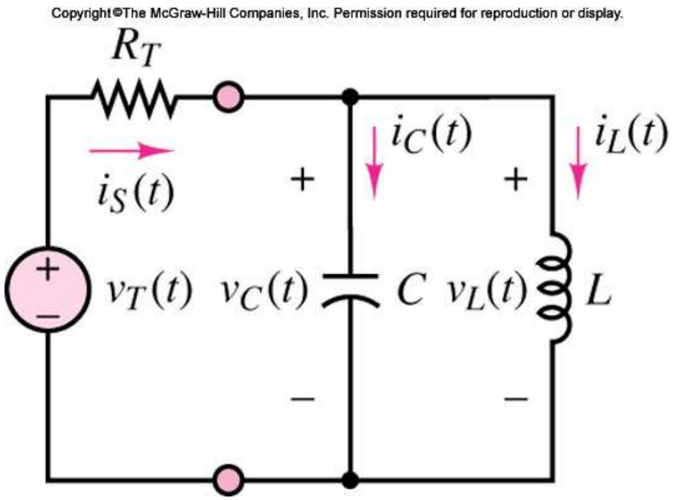


Note: Capacitor voltage is continuous,
but capacitor current is not (many jumps).

$$\frac{[x(t=\tau)-x(t=0)]}{[x(t=\infty)-x(t=0)]}=1-e^{-1}=0.632$$

Second Order Transient Response

- Second-order circuit: two energy storage element w/wo one energy loss element (e.g. *RLC* circuit, *LC* circuit)

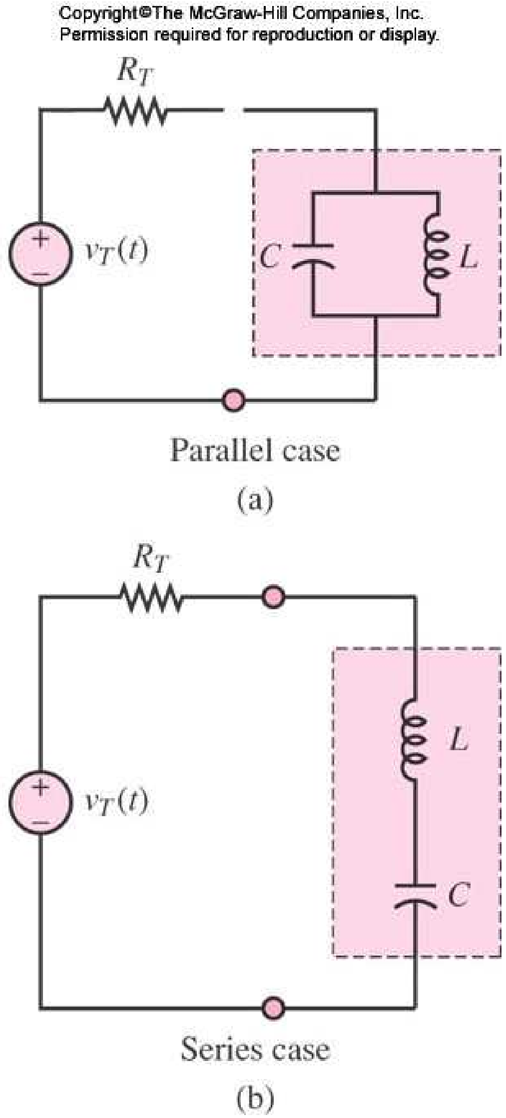


$$\text{KCL}: i_S = i_C + i_L \rightarrow \frac{v_R}{R_T} = i_L + i_C$$

$$\text{KVL}: -v_T + v_R + v_L = 0 \rightarrow v_R = v_T - v_L \text{ and } -v_T + v_R + v_C = 0 \rightarrow v_R = v_T - v_C$$

$$\frac{v_R}{R_T} = i_L + i_C \rightarrow \frac{1}{R_T}\left(v_T - L\frac{di_L}{dt}\right) = i_L + C\frac{dv_C}{dt} = i_L + C\frac{d}{dt}\left(L\frac{di_L}{dt}\right)$$

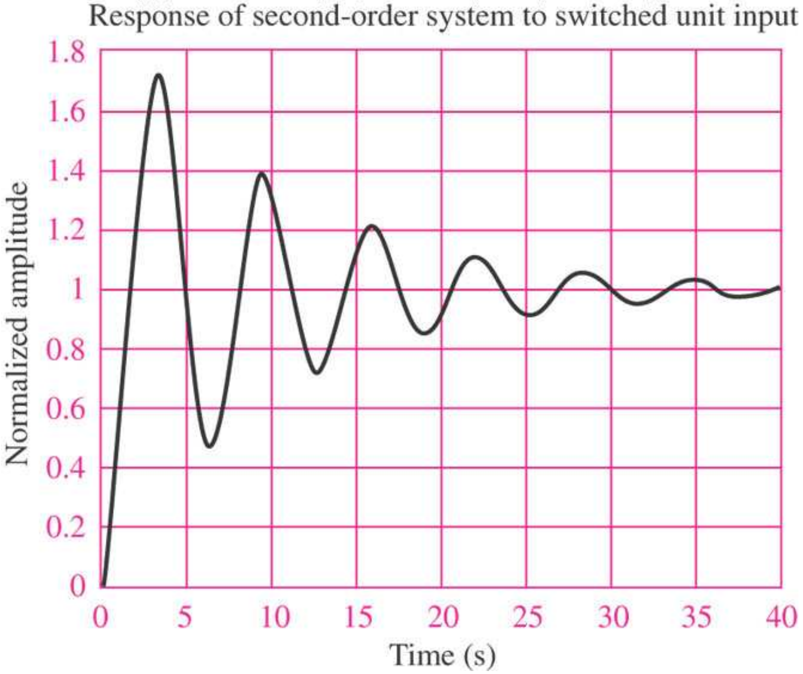
$$\frac{1}{R_T}\left(v_T - L\frac{di_L}{dt}\right) = i_L + LC\frac{d^2 i_L}{dt^2} \rightarrow \frac{v_T}{R_T} = LC\frac{d^2 i_L}{dt^2} + \frac{L}{R_T}\frac{di_L}{dt} + i_L$$



Second Order Transient Response (cont.)

$$a_2 \frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = b_0 f(t) \rightarrow \frac{1}{\omega_n^2}\frac{d^2x(t)}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx(t)}{dt} + x(t) = K_S f(t)$$

where the constants $\omega_n = \sqrt{a_0/a_2}$, $\zeta = (a_1/2)\sqrt{1/a_0 a_2}$ and $K_S = b_0/a_0$ termed the natural frequency, the damping ratio, and the DC gain, respectively.



$\omega_n = 1, \zeta = 0.1$ and $K_S = 1$

- The final value of 1 is predicted by the DC gain K_S=1, which tells us about the steady state.
- The period of oscillation of the response is related to the natural frequency w_n=1 leads to T=2 pi/w_n = 6.28 sec.
- The reduction in amplitude of the oscillation is governed by the damping ratio. With large damping ratio, the response not overshoots (oscillates) but looks like the first order response.
- Damping -> friction effect

Second Order Response

$$\frac{1}{\omega_n^2}\frac{d^2x(t)}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx(t)}{dt}+x(t)=K_S f(t)$$

Natural Response

$$\frac{1}{\omega_n^2}\frac{d^2x_N(t)}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx_N(t)}{dt}+x_N(t)=0$$

$\boxed{x_N(t)=\alpha_1 e^{s_1 t}+\alpha_2 e^{s_2 t}}$ where $s_{1,2}=-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$

Case 1: Real and distinct roots. $(\zeta>1) \rightarrow$ Overdamped response

$\rightarrow$ Look like the first order system

$$s_{1,2}=-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

Case 2: Real and repeated roots. $(\zeta=1)$

$\rightarrow$ Critically overdamped response $\rightarrow$ Oscillation

$$s_{1,2}=-\omega_n$$

Case 3: Complex roots. $(\zeta<1) \rightarrow$ Underdamped response $\rightarrow$ Oscillation

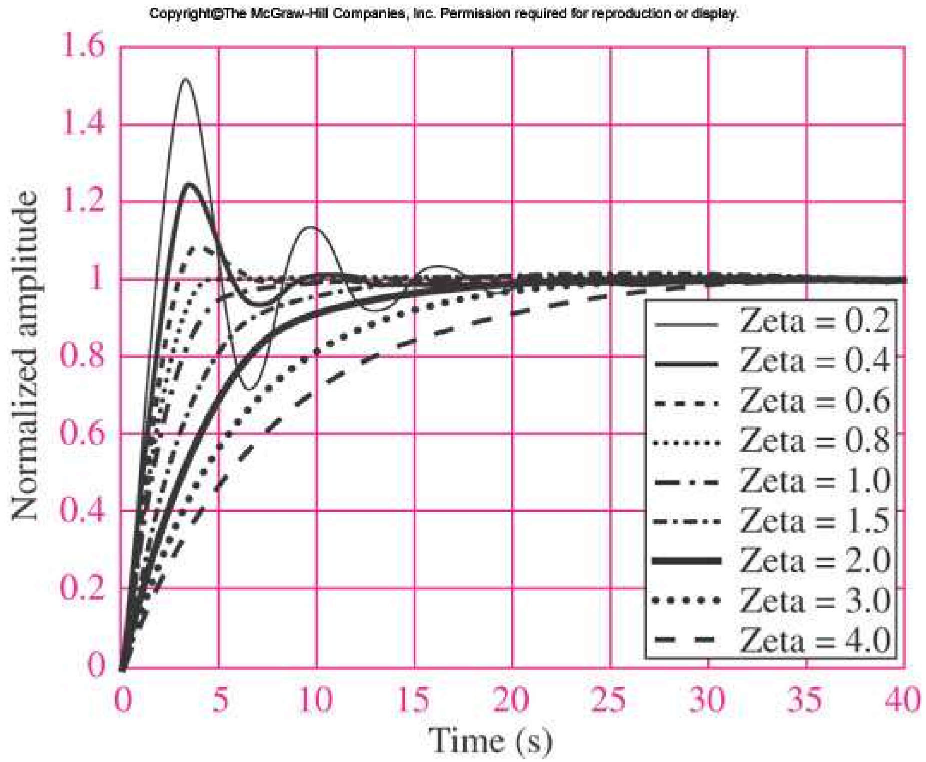
$$s_{1,2}=-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Forced Response due to DC (where $f(t)=F$): $\frac{dx_F(t)}{dt}\rightarrow 0$

$$\frac{1}{\omega_n^2}\frac{d^2x_F(t)}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx_F(t)}{dt}+x_F(t)=K_S f(t)\ \ t\geq 0 \rightarrow \boxed{x_F(t)=K_S F}\ \ t\geq 0$$

Complete Response

$x(t)=x_N(t)+x_F(t)$ α_1 and α_2 is constants that will be determined by the initial conditions.



Second Order Response (cont.)

- Procedures
  - Write the differential equation of the circuit for $t=0^+$, that is, immediately after the switch has changed. The variable x(t) in the differential equation will be either a capacitor voltage or an inductor current. You can reduce the circuit to Thevenin or Norton equivalent form. Rewrite the equation as the standard form.
  - Identify the initial conditions x(t=0⁺) and dx/dt(t=0⁺) using the continuity of capacitor voltages and inductor currents.
  - Write the complete solution for the circuit in the form.

$$\text{Case 1: Real and distinct roots.}(\zeta > 1): x(t) = \alpha_1 e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)t} + \alpha_2 e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)t} + x_F(t)$$

$$\text{Case 2: Real and repeated roots.}(\zeta = 1): x(t) = \alpha_1 e^{(-\omega_n)t} + \alpha_2 t e^{(-\omega_n)t} + x_F(t)$$

$$\text{Case 3: Complex roots.}(\zeta < 1): x(t) = \alpha_1 e^{\left(-\zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2}\right)t} + \alpha_2 e^{\left(-\zeta\omega_n - j\omega_n\sqrt{1 - \zeta^2}\right)t} + x_F(t)$$

  - Apply the initial conditions to solve for the constants α_1 and α_2 .

Example: Second Order Response

$$\text{Step1: KCL}: i_S = i_C = i_L \rightarrow \frac{v_R}{R_T} = i_L + i_C$$

$$\text{KVL}: -v_S + v_R + v_L + v_C = 0 \rightarrow v_R + v_L + v_C = v_S$$

$$i_L R + L\frac{di_L}{dt} + v_C(t=0) + \int_0^t \frac{i_L(t')}{C}dt' = v_S \rightarrow L\frac{d^2 i_L}{dt^2} + R\frac{di_L}{dt} + \frac{i_L}{C} = \frac{dv_S}{dt} = 0$$

$$\text{Step2: } v_C(t=0^-) = 5\text{ V} = v_C(t=0^+), i_L(t=0^-) = 0\text{ A} = i_L(t=0^+)$$

$$i_L(t=0^+)R + L\frac{di_L}{dt}(t=0^+) + v_C(t=0) = v_S \rightarrow 1\frac{di_L}{dt}(t=0^+) + 5\text{ V} = 25\text{V} \rightarrow \frac{di_L}{dt}(t=0^+) = 20\text{A/s}$$

$$\text{Step3: } L\frac{d^2 i_L}{dt^2} + R\frac{di_L}{dt} + \frac{i_L}{C} = 0 \rightarrow LC\frac{d^2 i_L}{dt^2} + RC\frac{di_L}{dt} + i_L = 0 : \frac{1}{\omega_n^2}\frac{d^2 x(t)}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx(t)}{dt} + x(t) = K_S f(t)$$

$$\frac{1}{\omega_n^2} = LC \rightarrow \omega_n = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{10^{-6}}} = 1000\,(\text{rad/s}), \frac{2\zeta}{\omega_n} = RC \rightarrow \zeta = \frac{RC\omega_n}{2} = \frac{R}{2}\sqrt{\frac{C}{L}} = \frac{5000}{2}\sqrt{\frac{10^{-6}}{1}} = 2.5$$

$\rightarrow$ Overdamped response

$$i_L(t) = \alpha_1 e^{s_1 t} + \alpha_2 e^{s_2 t} \quad \text{where } s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Complete Response (forced response = 0)

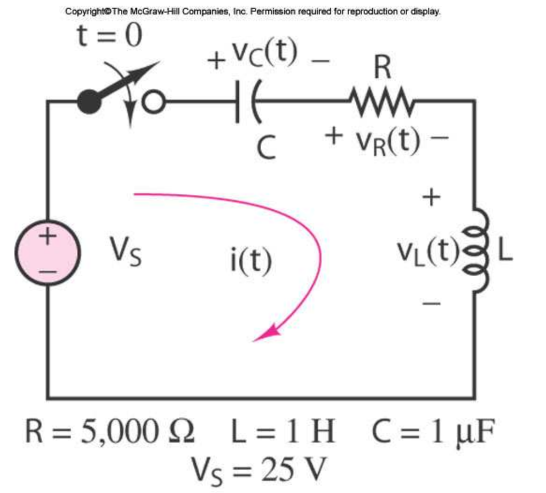
$$i_L(t) = \alpha_1 e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} + \alpha_2 e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t}$$

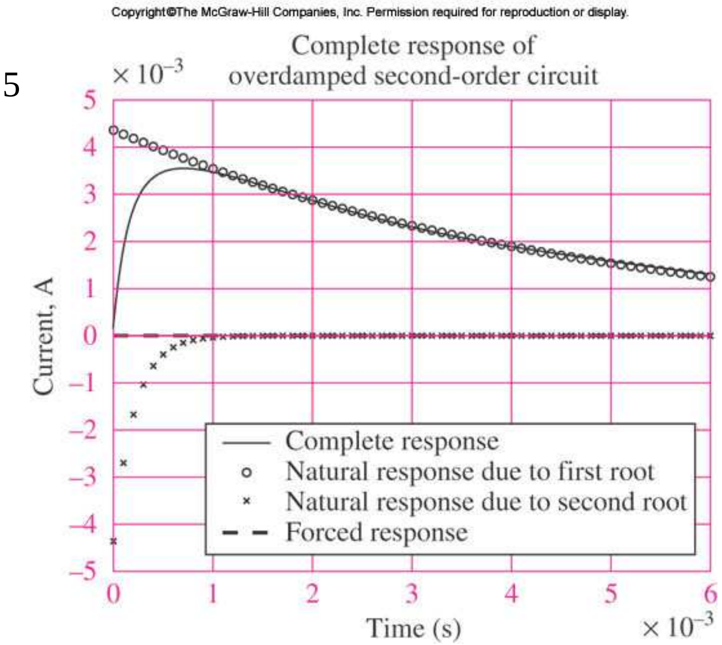
$$\text{Step4: Using } 0\text{ A} = i_L(t=0^+) \text{ and } \frac{di_L}{dt}(t=0^+) = 20\text{A/s, determine the constants } \alpha_1 \text{ and } \alpha_2$$

$$i_L(t=0^+) = 0 = \alpha_1 + \alpha_2$$

$$\frac{di_L}{dt} = \alpha_1\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} + \alpha_2\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t}$$

$$\frac{di_L}{dt}(t=0^+) = 20 = \alpha_1\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right) + \alpha_2\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)$$





Overdamped and Underdamped Circuit